

# On Batchelor passive advection by a finite-time correlated random velocity field

Stanislav A. Boldyrev

*Institute for Theoretical Physics, Santa Barbara, California 93106*

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The Batchelor passive advection is an advection by a smooth velocity field. If the velocity field is a  $\delta$ -correlated in time random Gaussian process, then the problem is reduced to quantum mechanics of fluctuating velocity gradient  $\frac{\partial u^i}{\partial x^j}(t)$ . For the finite-time correlated velocity field, such a reduction does not exist. To illustrate this point, the second moment of a passively advected magnetic field is considered, and the stochastic calculus is used to find finite-time corrections to its growth rate. The growth rate depends on large scale properties of the velocity field. Moreover, the problem is not universal with respect to the short-time regularization: different regularizations give different answers for the growth rate.

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## I. INTRODUCTION

Magnetic field passively advected by an incompressible velocity field obeys the induction equation

$$\partial_t B^i + v^k \partial_k B^i - B^k \partial_k v^i = \eta \Delta B^i, \quad (1)$$

where  $\eta$  is the diffusivity. We will assume that the diffusivity is negligibly small (large Prandtl number), and the magnetic field is therefore frozen into the fluid. The incompressibility of the velocity field is not relevant for further calculations; for the compressible case, Eq. (1) holds for  $B^i/\rho$ , where  $\rho$  is the density of the fluid. The random velocity field is assumed to be Gaussian in the Eulerian frame:

$$\begin{aligned} \langle v^i(\mathbf{x}, t) \rangle &= 0, \\ \langle v^i(\mathbf{x}, t) v^j(\mathbf{x}', t') \rangle &= \kappa^{ij}(\mathbf{x} - \mathbf{x}', t - t'), \end{aligned} \quad (2)$$

where  $\kappa^{ij}(\mathbf{x}, t)$  is an arbitrary function regular at  $\mathbf{x} = 0$ , this is the so-called Batchelor regime [1]. The solution of the Eq. (1) is

$$B^i(x, t) = B_0^i + \int_0^t \frac{\partial \tilde{v}^i(y, \tau)}{\partial y^k} d\tau B_0^k = B_0^i + \int_0^t \frac{\partial \tilde{v}^i(y(x, t), \tau)}{\partial x^k} d\tau B^k(x(y, t), t), \quad (3)$$

where  $y^i = x^i(y, 0)$  and  $B_0^i = B^i(y, 0)$  are initial values of the fluid-particle coordinate and of the magnetic field. The object  $\tilde{v}^i(y, \tau)$  in the integrand is the *Lagrangian* velocity field. The *Eulerian* velocity has the form  $v^i(x, \tau) = \tilde{v}^i(y(x, \tau), \tau)$ . To make use of the formula (2), we need to “stop” the moving point  $x^i = x^i(y, t)$  in the function  $\tilde{v}^i(y(x, \tau), \tau)$ . This is done differently in the Eulerian and Lagrangian frames (see Fig.1).

## II. LAGRANGIAN FRAME

In the Lagrangian frame,  $\mathbf{y}$  is fixed and we need to express  $\tilde{v}^i(y, \tau)$  in terms of  $v^i(y, \tau)$ . The general expression is  $\tilde{v}^i(y, \tau) = v^i\left(y^k + \int_0^\tau \tilde{v}^k(y, \tau'), \tau\right)$ . For small  $t$ , we expand this expression up to the second order in  $\int_0^\tau \tilde{v}^i(y, \tau')$ , and iterate once with respect to  $\tilde{v}$ . We obtain

$$\tilde{v}^i(\tau) = v^i(\tau) + v_l^i(\tau) \int_0^\tau v^l(\tau') + v_l^i(\tau) \int_0^\tau v_j^l(\tau') \int_0^{\tau'} v^j(\tau'') + \frac{1}{2} v_{lj}^i(\tau) \int_0^\tau v^l(\tau') \int_0^\tau v^j(\tau'') + \dots, \quad (4)$$

where all the velocities are taken at the point  $\mathbf{y}$ , and we use the short-hand notation  $v_l^i(\tau) \equiv \partial v^i(y, \tau)/\partial y^l$  and  $v_{lj}^i(\tau) \equiv \partial^2 v^i(y, \tau)/\partial y^l \partial y^j$ .

To obtain the equation for the second moment of  $B^i$ , we substitute the expansion (4) into the formula (3), raise the latter to the second power and average using formula (2). We assume that  $t$  is much smaller than the inverse growth rate of the second moment, and that the correlation time  $\tau_c$  is much smaller than  $t$ . Assuming isotropic and spatially homogeneous initial distribution,  $\langle B_0^i \rangle = 0$ ,  $\langle B_0^i B_0^j \rangle = \frac{1}{d} \delta^{ij} H_2(0)$ , we find the equation for the second-order moment  $H_2(t) = \langle |\mathbf{B}|^2 \rangle(t)$  in the Lagrangian frame. For the incompressible velocity field, it takes the form

$$\frac{d}{dt} H_2 = -\frac{1}{d} \kappa_{jj}^{ii} H_2 - \frac{1}{24d} \tau_c (\kappa_{jjlm}^{ii} \kappa^{lm} - 8 \kappa_{jl}^{im} \kappa_{jm}^{il} + 10 \kappa_{lm}^{ii} \kappa_{jj}^{lm}) H_2. \quad (5)$$

To derive this equation we assumed that  $\kappa(y, t) \simeq \kappa(y) T(t)$ , where  $T(t)$  is a  $\delta$  function smeared over the correlation time  $\tau_c$ . In particular, we have chosen the “box” regularization:  $T(t) = 1/\tau_c$  for  $t \in [-\tau_c/2, \tau_c/2]$ , and  $T(t) \equiv 0$  otherwise. The coefficients in the correction terms in the formula (5) are given by the following integrals:

$$I_1 = \frac{1}{2} \int_0^t \int_0^t T(\tau_1 - \tau_2) \int_{\tau_2}^{\tau_1} \int_{\tau_1}^{\tau_2} T(\tau' - \tau'') = -\frac{1}{24} t \tau_c + \dots, \quad (6)$$

$$I_2 = 2 \int_0^t \int_0^t \int_0^{\tau_1} \int_0^{\tau_2} T(\tau_2 - \tau') T(\tau_1 - \tau'') = \frac{1}{3} t \tau_c + \dots, \quad (7)$$

$$I_3 = \int_0^t \int_0^t T(\tau_1 - \tau_2) \int_0^{\tau_1} \int_0^{\tau_2} T(\tau' - \tau'') = \frac{t^2}{2} - \frac{5}{12} t \tau_c + \dots \quad (8)$$

All the derivatives of  $\kappa(y)$  in formula (5) are taken at  $y = 0$ ; the subscripts denote derivatives with respect to the corresponding components of  $\mathbf{y}$ , and  $d$  is the space dimension.

One can easily check that the first-order  $\tau_c$  correction to the growth rate is negative. It is important to note that this correction is not universal since the integrals (6-8) depend on the chosen regularization  $T(t)$  of the  $\delta$  function.

### III. EULERIAN FRAME

In the Eulerian frame, the point  $\mathbf{x}$  is fixed in the formula (3), and we need to express  $\tilde{v}^i(y(x, t), \tau)$  in terms of  $v^i(x, \tau)$ . By analogy with the expression (4), we write  $\tilde{v}^i(y, \tau) = v^i\left(x^k - \int_{\tau}^t \tilde{v}^k(y, \tau') d\tau', \tau\right)$ . The expansion now takes the form

$$\tilde{v}^i(\tau) = v^i(\tau) - v_l^i(\tau) \int_{\tau}^t v^l(\tau') d\tau' + v_l^i(\tau) \int_{\tau}^t v_j^l(\tau') \int_{\tau'}^t v^j(\tau'') d\tau'' + \frac{1}{2} v_{lj}^i(\tau) \int_{\tau}^t v^l(\tau') \int_{\tau}^t v^j(\tau'') d\tau'' + \dots, \quad (9)$$

where we use the short-hand notation  $v_l^i(\tau) \equiv \partial v^i(x, \tau) / \partial x^l$ . To calculate the second moment of  $B^i$ , we have to use the expression (3):

$$B^i(x, t) = B_0^k \left( \delta_k^i - \int_0^t \frac{\partial \tilde{v}^i(y, \tau)}{\partial x^k} d\tau \right)^{-1}, \quad (10)$$

that should be expanded up to the fourth order in  $\int_0^t \frac{\partial \tilde{v}^i(y, \tau)}{\partial x^k} d\tau$ . Then, we substitute the expansion (9) for  $\tilde{v}$ . Finally, assuming isotropic and spatially homogeneous initial distribution,  $\langle B_0^i \rangle = 0$ ,  $\langle B_0^i B_0^j \rangle = \frac{1}{d} \delta^{ij} H_2(0)$ , and making use of (2), we can find the equation for  $H_2(t)$ . We do not present the detailed calculation here since for the incompressible velocity field, the answer coincides with the Lagrangian case (5), as it should.

The appearance of the term  $\kappa_{jjlm}^{ii} \kappa^{lm}$  in Eq. (5) shows that for the finite correlation time  $\tau_c$ , the velocity field  $v^i(x, t)$  cannot be treated as linear in the calculation of the growth rate. This is obvious since a Lagrangian particle is swept by the finite distance  $v^i \tau_c$  before the velocity field becomes decorrelated. The appearance of  $\kappa(0)$  (or, equivalently,  $u_{rms}$ ) is a manifestation of the absence of the Galilean invariance. For the velocity field with finite correlation time, such invariance is broken as it can be seen from Eq. (2).

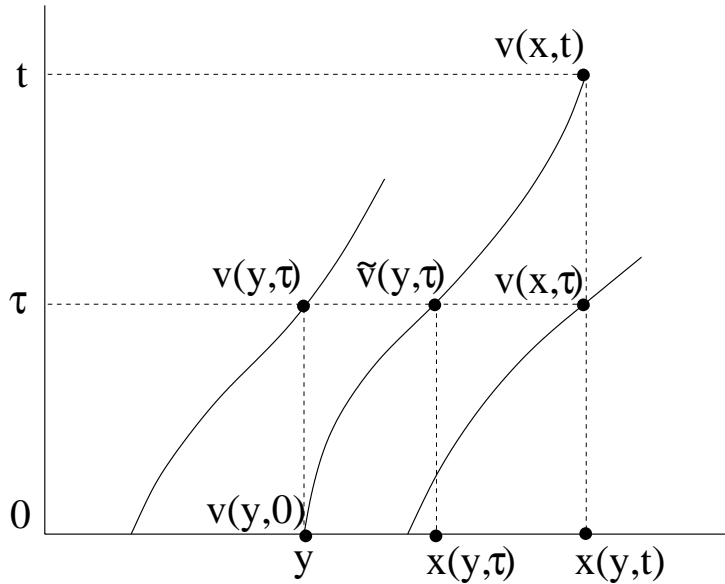


FIG. 1. Trajectories of Lagrangian particles.

## IV. DISCUSSION

The Batchelor advection, i.e. the advection by a smooth velocity field is a good model for many physical problems. It is usually a valid approximation when one is interested in the correlators of the advected fields on the scales  $l_{adv}$  that are much smaller than the viscous scale of the velocity field  $l_v$ .

An additional simplification arises when the correlation time of the velocity field  $\tau_c$  is much smaller than the inverse growth rates of the advected fields. Mathematically, such a limit is described by the  $\delta$ -correlated in time velocity field. In such a case, the growth rates depend only on  $\kappa''(0)$ , as a consequence of the scale invariance. In this case one can substitute the linear velocity field  $v^i(x, t) = \sigma_k^i(t)x^k$ ,  $\langle \sigma_l^i(t)\sigma_m^j(0) \rangle = \kappa_{lm}^{ij}(0)\delta(t)$ , for the real field, and thus reduce the field-theoretical problem to the quantum mechanics of the fluctuating matrix  $\sigma_k^i(t)$ ; for details and references, see [2–4].

We have demonstrated that the possibility of such a reduction is an artifact of the  $\delta$ -correlated velocity field, and have presented a simple method for calculating the finite-time corrections to the growth rates. The criterion of the applicability of the quantum mechanical reduction is not only  $l_{adv}/l_v \ll 1$ , but also  $\tau_c \partial v / \partial x \ll 1$ . For  $\tau_c \neq 0$ , the universal growth rates do not exist, they are determined by the statistics of the velocity field on the integral scale and by the form of the short-time regularization. This fact is sometimes overlooked in the literature; in the present note we have tried to clarify the question. The general formalism allowing to systematically find the finite-time corrections to the dynamo growth rates, has been developed in [5] by a different method. A simple discussion of the stochastic calculus applied to the dynamo problem can be found in [6].

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